2) Equations of Motion

2.1) Newton's laws

- **Law I:** Every body continues in a state of rest or uniform motion in a right line unless it is compelled to change that state by forces impressed on it.

- **Law II:** The change of motion is proportional to the motive force impressed and is made in the direction of the right line in which that force was impressed.

- **Law III:** To every action there is always an opposed and equal reaction; or the mutual action of two bodies upon each other are always equal, and directed to contrary parts.

2.2 Newton's second law

Newtons second law postulates a relation between the acceleration of the body and forces acting on it.
We can reformulate newton's second law as follows:

Newton II:
The net force, $\vec{F}$, on a body of constant mass m causes the body to accelerate.
The acceleration $\vec{\ddot{x}}$ is in the direction of the force, proportional to the magnitude of the force and inversely proportional to the mass of the body

$$\boxed{\vec{\ddot{x}} = \frac{1}{m}\vec{F}}$$ or equivalently

$$\boxed{\vec{F} = m\vec{\ddot{x}}}$$

The previous equation is known as the <u>equation of motion</u>.

<u>Note that</u> :

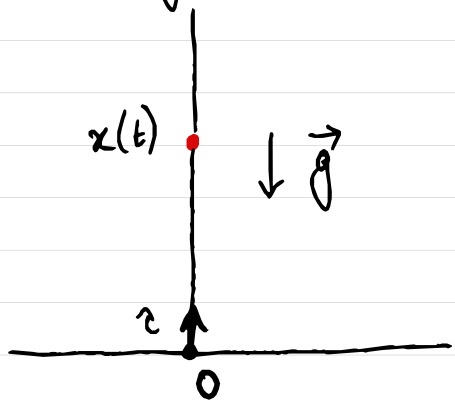
- <u>force</u> has <u>magnitude and direction</u> (so that it is <u>described by a vector</u>)
- there may be a number of forces acting on the body but the <u>acceleration is proportional</u> to the <u>net force</u>
- "<u>inversely proportional to the mass</u>" implies that the same force has a stronger impact on a smaller mass cause it has higher acceleration

2.3) Examples of Forces

• Weight (or uniform gravity force):

$$\vec{F} = m\vec{g}$$

$\vec{g}$ is the gravitational acceleration



$$\vec{F} = m\vec{g} \Rightarrow m\ddot{\vec{x}} = m\vec{g}$$

$$\Rightarrow \cancel{m}\ddot{x}\hat{\imath} = -\cancel{m}g\hat{\imath}$$

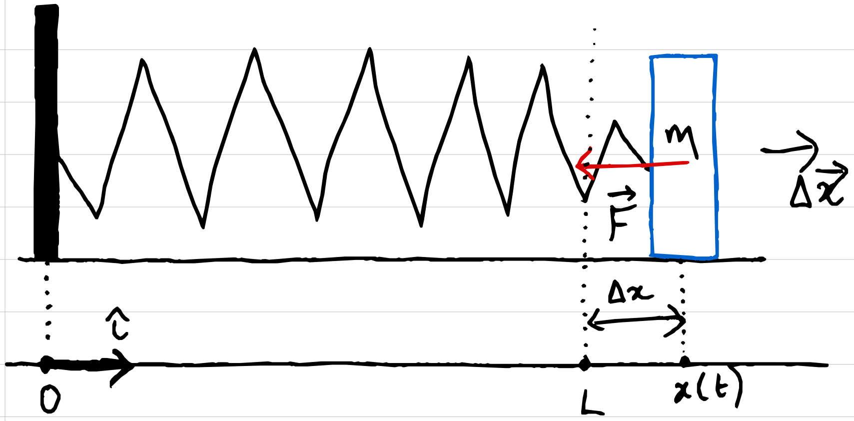
$$\Rightarrow \ddot{x}\hat{\imath} = -g\hat{\imath}$$

[$\vec{g}$ is in opposite direction to $\hat{\imath}$ $\Rightarrow \vec{g} = -g\hat{\imath}$]

which gives scalar equation

$$\ddot{x} = -g \Rightarrow \frac{d^2x}{dt^2} = -g$$

- Elastic force (coiled spring):



$$\vec{F} = -K\Delta\vec{x} \Rightarrow \vec{F} = -K\Delta x\hat{i}$$

[$\Delta\vec{x}$ is in the same direction to $\hat{i}$]

- Stokes friction:

$$\vec{F} = -\Gamma\vec{v} \Rightarrow \vec{F} = -\Gamma\dot{x}$$

(here Γ is a constant coefficient)

Note:

- In the weight example, the force is in the same direction as acceleration $\vec{g}$ so

$$\vec{F} = m\vec{g}$$

is the vector equation

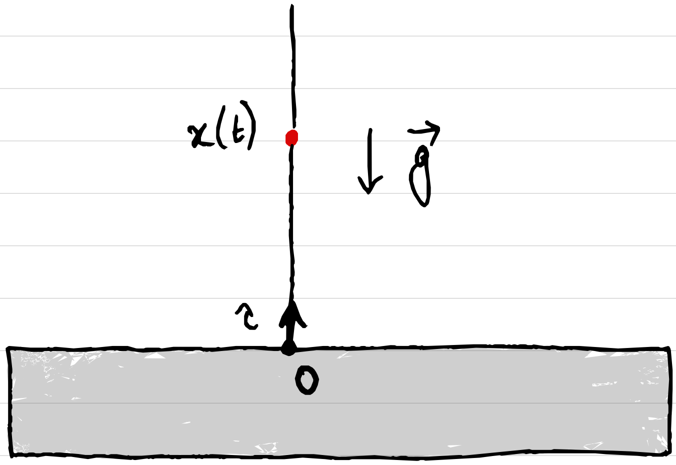
- In the elastic coil example, Force is in the opposite direction to the displacement so the vector equation is

$$\vec{F} = -K\Delta\vec{x}$$

This is known as Hooke's law where K is the coefficient of elasticity

2.4) Examples of Equation of Motion

- Vertical Motion under uniform gravity with stokes friction



$$\vec{F} = m\vec{g} - \Gamma v \Rightarrow m\ddot{x} = m\vec{g} - \Gamma \dot{x}$$

(note stokes friction is always negative since it always acts opposite to net force)

Let us choose a co-ordinate axis that is vertical and directed up with the origin at the ground level.

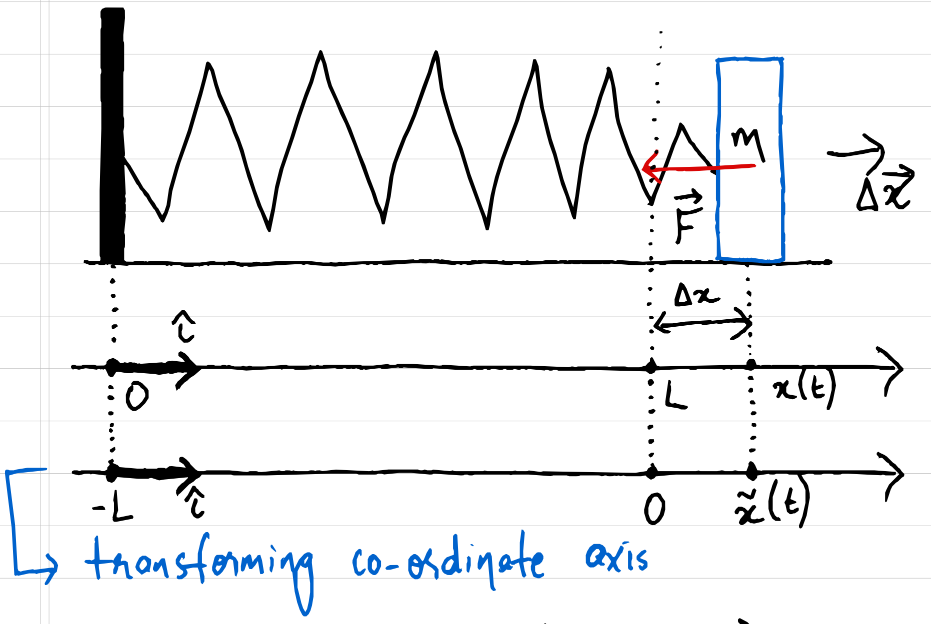
The gravity force is directed vertically down. This means that relative to the chosen co-ordinate axis

$$m\ddot{x}\hat{\imath} = -mg\hat{\imath} - \Gamma\dot{x}\hat{\imath}$$

where g is the gravitationall acceleration. This can be rewritten as the scalar equation:

$$m\ddot{x} = -mg - \Gamma\dot{x}$$

• Motion under an elastic force (Hooke's law):



Let the left end of a (weightless) coiled spring of natural length L be fixed (attached to a wall) and a body of mass m be attached to the right end of the spring and lie on a flat smooth surface. (there is no friction between body and the surface.

Then we introduce the co-ordinate axis x such that it is parallel to the surface with the origin at the wall. The force is given by

$$\vec{F} = -k\Delta\vec{x}$$

So equation of motion is

$$m\ddot{\vec{x}} = -k\Delta\vec{x}$$

Here $\Delta\vec{x}(t)$ is the displacement of the body from the equilibrium position (where length of spring is equal to its natural length L) to its current position

$$\Delta\vec{x}(t) = \vec{x}(t) - L\hat{\imath} \quad \text{or}$$

$$\Delta\vec{x}(t) = (x(t) - L)\hat{\imath}$$

So

$$m\ddot{x} = -k(x - L) \quad \text{or equivalently}$$

$$m\ddot{x} + kx = kL$$

Note that this is a linear inhomogeneous ODE but can be made homogeneous by introducing new variable

$$\tilde{x} = x - L$$

$$\tilde{x} = x - L \Rightarrow \ddot{\tilde{x}} = \ddot{x}$$

So

$$m\ddot{\tilde{x}} = -k\tilde{x}$$

Note that the new variable $\tilde{x}$ can be interpreted as the position of the body relative to a new co-ordinate axis $\tilde{x}$ which is parallel to the old one and shifted by length L to the right relative to the origin of the x axis.

↳ From this we can see that equations of motion can be simplified by choosing the right co-ordinate axis.

2.5) A free particle

When the net force is 0, the equations of motion reduces to

$$m\ddot{\vec{x}} = \vec{0} \quad \text{or} \quad \ddot{\vec{x}} = \vec{0}$$

$$\Rightarrow \quad \ddot{x} = 0$$

Integrating:

$$\ddot{x} = 0 \Rightarrow \frac{d^2 x(t)}{dt^2} = 0$$

$$\Rightarrow \frac{dx}{dt} = A$$

$$\Rightarrow \int dx = \int A\, dt$$

$$\Rightarrow \quad x = At + B$$

constants

↳ found by initial conditions

Let initial conditions be $\dot{x}(0) = v_0$, $x(0) = x_0$

So

$$x = v_0 t + x_0$$

This means the particle moves with a constant velocity (if $v_0 \neq 0$) or is at rest $x = x_0$ ($v_0 = 0$)

This agrees with newtons first laws

Defn: **Momentum:**
The quantity

$$p(t) = m\dot{x}(t)$$

is called the momentum of the particle

In terms of momentum,

$$\vec{F} = m\ddot{\vec{x}}(t) = \dot{\vec{p}}(t)$$

So force is <u>rate of change of momentum</u>

$$\vec{F} = m\ddot{\vec{x}}(t) = \dot{\vec{p}}(t)$$

$$\Rightarrow \quad m\ddot{x}(t) = \dot{p}(t) \quad \text{(scalar eqn).}$$

In terms of momentum, eqn of free particle becomes

$$\dot{p} = 0$$

which means p does not change with time i.e.

$$p(t) = p(0) \quad \forall t > 0$$

Defn: **Constants of Motion:**

Quantities which do not change with time are called <u>conserved quantities</u> or <u>constants of motion</u> in mechanics.

Thus momentum of a free particle is a constant of motion.

↳ The fact is also referred to as "<u>law of conservation of momentum</u>"

Multiplying equation of a free particle by $\dot{x}$, we obtain

$$m\ddot{x} = 0 \Rightarrow m\ddot{x}\dot{x} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{m\dot{x}^2}{2}\right) = 0 \qquad \text{[chain rule]}$$

Defn: <u>**Kinetic energy:**</u>

The <u>kinetic energy</u> is defined by

$$T = \frac{m\dot{x}^2}{2}$$

Therefore T is a constant of motion for a free particle

This <u>fact</u> is a <u>particular</u> case of "<u>law of conservation of energy</u>" for <u>this system</u>